AN ALGORITHM FOR THE SOLUTION OF LINEAR PROGRAMING PROBLEMS

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LINEAR PROGRAMMING PROBLEMS

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ABSTRACT

Linear programming techniques are becoming of greater importance because the use of computerization has increased the fields for applications for linear programs. The primal-dual algorithm, in which the constraints are added one at a time, is investigated as a possible faster solution method. A computer program was developed to compare this method with the standard primal-dual algorithm using the full set of constraints at one time. Several random problems were solved using these two methods, and the results indicated a significant improvement in the solution time by the use of adding the constraints one at a time.

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I. INTRODUCTION

New linear programming algorithms have been developed to reduce the computational time in solving linear programs. The purpose of this thesis is to investigate the merits of one such new algorithm. This method consists of introducing the constraint equations one at a time. After each constraint is added, the "smaller", or submatrix, problem is solved using the primal-dual algorithm. This continues until all constraints have been added and a solution is obtained.

The rationale for this approach is that small matrices are used in the initial stages of solving the linear program; the size of the matrices increases only when additional constraints are introduced. If the number of iterations used in this method is not significantly different from the number of iterations used with the full matrices, the manipulation of the smaller matrices in the initial stages will reduce the solution time.

II. NOTATION

m number of constraint equations.

n number of legitimate variables.

A mxn matric of coefficients of the constraint equations with elements a_{ij}, $i = 1,\ldots,m$, $j = 1,\ldots,n$.

P mxm matrix of the basis vectors.

P^{-1} inverse of the basis.

P_j mxl column vector which is the j^{th} column of A.

$$\text{i.e., } P_j = \begin{bmatrix} a_{ij} \\ \cdot \\ \cdot \\ \cdot \\ a_{mj} \end{bmatrix}, \; j = 1,\ldots,n \; .$$

P_{ai} mxl column vector associated with the i^{th} artificial variable, $i = 1,\ldots,m$.

x_j mxl column vector with elements x_{ij}, where $X_j = P^{-1}P_j$.

C nxl column vector with elements c_j which are the costs of the legitimate variables.

B mxl column vector with elements b_i which are the right-hand sides of the constraint equations.

s_j dual slack variables.

$\hat{s}_j$ dual slack variables after a dual iteration.

$\bar{d}$ mxl column vector whose elements are the coefficients of the basis variables of the added constraint equation.

Notation used with tableau:

P	B	P_0	P_1	P_2	P_3	P_4	P_5	P_{a0}	P_{a1}	
P_{a0}	b_0-4	1	1/2	1/3	0	5/6	1	1	-1/6	$\theta = b_0 - 4$
P_3	4	0	1/2	2/3	1	1/6	0	0	1/6	
z_j-c_j	$4-b_0$	$\boxed{-1}$	-1/2	-1/3	0	-5/6	-1	-------		$\hat{\theta} = -4/-1 = 4$
s_j	$4b_0$	4	2	4	0	4	4	-------		
$\hat{s}_j$	16	0	0	8/3	0	2/3	0	-------		
		↑								

↑ the vector to be introduced into the basis. e.g., in this tableau P_0 will be introduced.

1 pivot element for a primal iteration, i.e., the θ criterion is $\theta = \min_i (x_{iB}/x_{ij})$ such that $x_{ij} > 0$.

$\boxed{-1}$ pivot element for a dual iteration, i.e., the $\hat{\theta}$-criterion is $\hat{\theta} = \min_j (-s_j/z_j-c_j)$ such that $z_j-c_j < 0$.

III. FORMULATION OF THE PROBLEM AND SOLUTION PROCEDURE

The general linear programming problem is to maximize

$$z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \geq 0, \; i = 1,\ldots,m \; , \qquad (1)$$

and

$$x_j \geq 0, \; j = 1,\ldots,n \; .$$

The <u>modified primal</u> uses an additional constraint

$$x_0 + \sum_{j=1}^{n} x_j = b_0 ,$$

where the cost of x_0 is zero and b_0 is arbitrarily large, so that for $x_0 > 0$ the constraint adds no additional restriction on (1).

The modified primal is written to maximize

$$z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\left.\begin{aligned} x_0 + \sum_{j=1}^{n} x_j &= b_0 \; , \\ \sum_{j=1}^{n} a_{ij} x_j &= b_i , \; i = 1,\ldots,m \; , \end{aligned}\right. \qquad (2)$$

and

$$x_j \geq 0, \; j = i,\ldots,n \; .$$

From (2) we can write the <u>modified</u> <u>dual</u> with slack variables, s_j, $j = 0,1,\ldots,n$, added. That is to minimize

$$w_0 b_0 + \sum_{i=1}^{m} w_i b_i$$

subject to

$$w_0 \qquad\qquad - s_0 = 0,$$

$$w_0 + \sum_{i=1}^{m} w_i a_{ij} - s_j = c_j,\ j = 1,\ldots,n, \tag{3}$$

and

$$w_i \text{ unrestricted for } i = 0,1,\ldots,m.$$

The starting feasible solution to the primal dual algorithm is $w_i = 0$, $i = 1,\ldots,m$, and $w_0 = \max_j (c_j, 0)$.

For an optimal solution the complementary slackness condition must hold. That is

$$s_0 x_0 + \sum_{j=1}^{n} s_j x_j = 0\ . \tag{4}$$

Adding artificial variables, x_{ai}, $i = 0,1,\ldots,m$, to (2) with the cost of the artificial variables set to -1 and the cost of the legitimate variables set to zero, the <u>extended primal</u> can be written as

maximize

$$- x_{a0} - \sum_{i=1}^{m} x_{ai}$$

subject to

$$x_0 + \sum_{j=1}^{n} x_j + x_{a0} \quad = b_0\ ,$$

$$\sum_{j=1}^{n} a_{ij} x_j + x_{ai} = b_i,\ i = 1,\ldots,m,$$

and

$$x_j \geq 0,\ j = 1,\ldots,n, \text{ and } x_{ai} \geq 0,\ i = 0,\ldots,m\ .$$

Solving the extended primal is similar to using a Phase I Revised Simplex method.[1] However, in the primal-dual algorithm, when Phase I ends the linear program is solved because complementary slackness is maintained throughout the solution procedure.

If a new constraint is added to the tableau, complementary slackness is maintained without changing the dual slack variables. This can be shown as follows:

> Assume we have a feasible solution to the problem with k constraint equations. This means that $s_j = 0$ for all j such that $P_j \in P$ and $x_j = 0$ for all j such that $P_j \notin P$. These conditions imply that complementary slackness is maintained, and that the modified dual also has a feasible solution.
>
> Now we add the $k + 1^{st}$ constraint which introduces a new dual variable, w_{k+1}, but no new dual slack variables, s_j, $j = 0,1,\ldots,n$. We need a feasible solution to the modified dual for the enlarged system. Observe that we have a feasible solution if we set $w_{k+1} = 0$ since then the s_j, $j = 0,1,\ldots,n$, remain unchanged. In particular, $s_j = 0$ for all j such that $P_j \in P$, that is, for the legitimate variables. Also, $x_j = 0$ for all j such that $P_j \notin P$, which implies that we have maintained complementary slackness.

It is worth noting that the $z_j - c_j$, $j = 0,1,\ldots,n$, must be recalculated since the new constraint which is added to the extended primal starts with its artificial variable, $x_{a,k+1}$, in the basis with its cost set at -1.

The new basis is $\bar{P} = \begin{matrix} P & \bar{0} \\ \bar{d}^T & 1 \end{matrix}$, where $P_{a,k+1} = \begin{matrix} \bar{0} \\ 1 \end{matrix}$ is the artificial vector associated with $x_{a,k+1}$. Now we can solve the new k + 1 system using the primal-dual algorithm since complementary slackness has been maintained.

An optimal solution exists if and only if the following criteria are satisfied:

1. $z_j - c_j \geq 0$ for $j = 0,1,\ldots,n$,
2. $z_B - c_B = 0$, and
3. $x_0 > 0$.

The solution procedure is as follows:

The first tableau is set up using the first two constraints of the <u>extended primal</u> and the starting solution to the modified dual, which implies that at least one $s_j = 0$.

<u>Step 1</u>. Is there a j, say j_0, such that $s_{j0} = 0$ and $z_{j0} - c_{j0} < 0$?

a. Yes. Go to 2.

b. No. Go to 3.

<u>Step 2.</u> Introduce P_{j0} into the basis using the minimum θ-criterion and a primal iteration. Since the extended primal is bounded a pivot will always exist. Note that the s_j remain unchanged for all j. Go to 1.

<u>Step 3</u>. Is $z_j - c_j < 0$ for some j?

a. Yes. Go to 4.

b. No. Go to 5.

Step 4. Use the minimum $\hat{\theta}$-criterion. Is $\hat{\theta}$ bounded?

a. Yes. Perform a dual iteration to compute a new set of s_j's, say $\hat{s}_j$'s. Go to 1.

b. No. The linear program has no feasible solution. Stop.

Step 5. Is $z_B - c_B < 0$?

a. Yes. The linear program has no feasible solution. Stop.

b. No. Go to 6.

Step 6. Have all of the constraints been added?

a. Yes. Go to 9.

b. No. Go to 7.

Step 7. Introduce the next restraint, say the $k + 1^{st}$. Place the artificial vector $P_{a,k+1}$ in the basis. Compute $x_{B,k+1}$. Is $x_{B,k+1} \geq 0$?

a. Yes. Go to 8.

b. No. Multiply all coefficients of the new constraint, except for the artificial variable, by -1. This assures that $x_{B,k+1} \geq 0$ and the artificial variable is non-negative. Go to 8.

Step 8. For the system with $k + 1$ restraints, compute the new values of $z_j - c_j$ for $j = 0,1,\ldots,n$. Go to 1.

Step 9. Is $x_0 = 0$?

a. Yes. The linear program is unbounded. Stop.

b. No. An optimal solution has been found. Stop.

IV. SAMPLE PROBLEM

Consider the following example:

maximize

$$z = 2x_1 + 4x_3$$

subject to

$$\begin{aligned} 3x_1 + 4x_2 + 6x_3 + x_4 \quad &= 24\ , \\ 4x_1 + 3x_2 + 12x_3 \quad + x_5 &= 24\ , \\ x_1 + x_2 + 4x_3 \quad &= 8\ , \end{aligned}$$

and

$$x_j \geq 0,\ j = 1,\ldots,5.$$

Then the extended primal is

maximize

$$- x_{a0} - x_{a1} - x_{a2} - x_{a3}$$

subject to

$$\begin{aligned} x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_{a0} \quad &= b_0, \\ 3x_1 + 4x_2 + 6x_3 + x_4 \quad + x_{a1} \quad &= 24, \\ 4x_1 + 3x_2 + 12x_3 \quad + x_5 \quad + x_{a2} \quad &= 24, \\ x_1 + x_2 + 4x_3 \quad + x_{a3} &= 8, \end{aligned}$$

$x_j \geq 0$ for $j = 0,\ldots,5$, and $x_{ai} \geq 0$ for $i = 0,\ldots,3$.

The dual slack variables are $s_0 = \max_j(c_j,0) = 4$ with $j = 3$.

Then $s_B = x_0 b_0 = 4b_0$, and $s_j = s_0 - c_j$ for $j = 1,\ldots,5$, so that $s_1 = 2$, $s_2 = 4$, $s_3 = 0$, $s_4 = 4$, and $s_5 = 4$.

The starting tableau, using the first original constraint, is

P	B	P_0	P_1	P_2	P_3	P_4	P_5	P_{a0}	P_{a1}	
P_{a0}	b_0	1	1	1	1	1	1	1	0	
P_{a1}	24	0	3	4	6	1	0	0	1	$\theta = 24/6 = 4$
z_j-c_j	$-b_0-24$	-1	-4	-5	$\boxed{-7}$	-2	-1	---	---	
s_j	$4b_0$	4	2	4	0	4	4	---	---	
					↑					

From step 1, we see that $s_3 = 0$ and $z_3-c_3 < 0$. Using the minimum θ-criterion (as discussed in section III) in step 2, we introduce P_3 into the basis and remove P_{a1} from the basis.

Since there is no j_0 for which $s_{j0} = 0$ and $z_{j0}-c_{j0} < 0$, but $z_j-c_j < 0$ for several values of j, we arrive at step 4. Using the minimum $\hat{\theta}$-criterion a new set of s_j's, called $\hat{s}_j$'s are calculated.

P	B	P_0	P_1	P_2	P_3	P_4	P_5	P_{a0}	P_{a1}	
P_{a0}	b_0-4	1	1/2	1/3	0	5/6	1	1	-1/6	$\theta = b_0-4$
P_3	4	0	1/2	2/3	1	1/6	0	0	1/6	
z_j-c_j	$-b_0+4$	$\boxed{-1}$	-1/2	-1/3	0	-5/6	-1	---	---	$\hat{\theta} = -4/-1 = 4$
s_j	$4b_0$	4	2	4	0	4	4	---	---	
$\hat{s}_j$	16	0	0	8/3	0	2/3	0	---	---	
		↑								

Now $\hat{s}_0 = 0$ and $z_0-c_0 < 0$ so, from step 2, we introduce P_0 into the basis and remove P_{a0} from the basis.

P	B	P_0	P_1	P_2	P_3	P_4	P_5	P_{a0}	P_{a1}
P_0	b_0-4	1	1/2	1/3	0	5/6	1	1	-1/6
P_3	4	0	1/2	2/3	1	1/6	0	0	1/6
z_j-c_j	0	0	0	0	0	0	0	---	---
s_j	16	0	0	8/3	0	2/3	0	---	---

From the above tableau we trace through steps 1b, 3b, 5b, 6b, and arrive at step 7. Note that we have obtained an optimal solution to the subproblem with one constraint. In step 7 we introduce the second constraint. Since P_0 and P_3 are basis vectors, $\bar{d}^T = (a_{20}, a_{23}) = (0,12)$; the new basis consists of P_0, P_3 and P_{a2}. With this basis we find that $x_{B2} = -24 < 0$ so that step 7b must be used. The second restraint is replaced by

$$- 4_{x1} - 3_{x2} - 12_{x3} - x_5 + x_{a2} = -24 \ ,$$

which is used throughout the remainder of the solution procedure. Note that now $\bar{d}^T = (0,-12)$ and $x_{B2} = 24 > 0$. New values of x_j-c_j for $j = 0,1,\ldots,n$ are computed (step 8), and the new tableau is

P	B	P_0	P_1	P_2	P_3	P_4	P_5	P_{a0}	P_{a1}	P_{a2}	
P_0	b_0-4	1	1/2	1/3	0	5/6	1	1	-1/6	0	
P_3	4	0	1/2	2/3	1	1/6	0	0	1/6	0	$\theta = \frac{4}{1/2} = 8$
P_{a2}	24	0	2	5	0	2	-1	0	2	1	
z_j-c_j	-24	0	$\boxed{-2}$	-5	0	-2	1	---	---	---	
s_j	16	0	0	8/3	0	2/3	0	---	---	---	

Since $s_1 = 0$ and $z_1 - c_1 < 0$ (step 1) we go to step 2. Using a primal iteration, we introduce P_1 into the basis and eliminate P_3 from the basis.

P	B	P_0	P_1	P_2	P_3	P_4	P_5	P_{a0}	P_{a1}	P_{a2}	
P_0	$b_0 - 8$	1	0	-1/3	-1	2/3	1	1	-1/3	0	
P_1	8	0	1	4/3	2	1/3	0	0	1/3	0	
P_{a2}	8	0	0	7/3	-4	4/3	-1	0	4/3	1	$\theta = \frac{8}{4/3} = 6$
$z_j - c_j$	-8	0	0	-7/3	4	$\boxed{-4/3}$	1	---	---	---	$\hat{\theta} = \frac{-2/3}{-4/3} = 1/2$
s_j	16	0	0	8/3	0	2/3	0	---	---	---	
$\hat{s}_j$	12	0	0	3/2	2	0	1/2	---	---	---	
						↑					

Now $z_j - c_j \geq 0$ for all j for which $s_j = 0$. From steps 1b, 3a, and 4a, a new set of s_j's are computed. Then $\hat{s}_4$ becomes zero and $z_4 - c_4 < 0$ so, from step 2, P_4 enters the basis and P_{a2} is removed from the basis.

P	B	P_0	P_1	P_2	P_3	P_4	P_5	P_{a0}	P_{a1}	P_{a2}
P_0	$b_0 - 12$	1	0	-3/2	1	0	3/2	1	-1	-1/2
P_1	6	0	1	3/4	3	0	1/4	0	0	-1/4
P_4	6	0	0	7/4	-3	1	-3/4	0	1	3/4
$z_j - c_j$	0	0	0	0	0	0	0	---	---	---
s_j	12	0	0	3/2	2	0	1/2	---	---	---

Optimality has now been obtained with the second restraint added. Following steps 1b, 3b, 5b, 6b, and 7, we introduce the third and final restraint. Since the basis vectors were P_0, P_1, and P_4, $\bar{d}^T = (a_{30}, a_{31}, a_{34}) = (0,1,0)$. The new basis vectors are P_0, P_1, P_4, P_{a3}. We find that $x_{B3} = 2 > 0$ so we go to step 8 and recompute $z_j - c_j$ for $j = 0,1,\ldots,n$.

The next sequence of steps is 1b, 3a, and 4a, which leads to a new set of s_j's.

P	B	P_0	P_1	P_2	P_3	P_4	P_5	P_{a0}	P_{a1}	P_{a2}	P_{a3}	
P_0	b_0-12	1	0	-3/2	1	0	3/2	1	-1	-1/2	0	
P_1	6	0	1	3/4	3	0	1/4	0	0	-1/4	0	
P_4	6	0	0	7/4	-3	1	-3/4	0	1	3/4	0	
P_{a3}	2	0	0	1/4	1	0	-1/4	0	0	1/4	1	$\theta = 2$
z_j-c_j	-2	0	0	-1/4	$\boxed{-1}$	0	1/4	---	---	---	---	$\theta = -2/-1 = 2$
s_j	12	0	0	3/2	2	0	1/2	---	---	---	---	
$\hat{s}_j$	8	0	0	1	0	0	1	---	---	---	---	
					↑							

Steps 1a and 2 bring P_3 into the basis with the elimination of P_{a3} from the basis. The new tableau is:

P	B	P_0	P_1	P_2	P_3	P_4	P_5	P_{a0}	P_{a1}	P_{a2}	P_{a3}
P_0	b_0-14	1	0	-7/4	0	0	7/4	1	-1	-3/4	-1
P_1	0	0	1	0	0	0	1	0	0	- 1	-3
P_4	12	0	0	5/2	0	1	-3/2	0	1	3/2	3
P_3	2	0	0	1/4	1	0	-1/4	0	0	1/4	1
z_j-c_j	0	0	0	0	0	0	0	---	---	---	---
s_j	8	0	0	1	0	0	1	---	---	---	---

This tableau is the final tableau since the sequence of steps 1b, 3b, 5b, 6a, and 9b inform us that we have found an optimal solution to the original linear program. Note that the complementary slackness condition has been maintained, that is, $s_0x_0 + \sum_{j=1}^{n} s_jx_j = 0$. Therefore, the optimal solution is

$$x_0 = b_0 - 14 > 0,$$

$$x_1 = x_2 = x_5 = 0,$$

$$x_3 = 2, \text{ and}$$

$$x_4 = 12,$$

with the optimal cost $z = s_B = 8$.

V. PROGRAMMING TECHNIQUE

The linear programming technique described in this thesis was programmed in FORTRAN IV for use on the IBM 360/67 computer. One subroutine is used for the primal simplex iteration and another is used for the dual iteration. The final subroutine is used for the addition of constraints. The main (driving) routine is used to solve both the full linear program and the linear program using addition of constraints as described in this thesis. Using the main routine to solve both problems eliminates any time differences due to differences in programming techniques. Both of the solution procedures were timed*, and the number of iterations of each were counted. Read and print times were not included in the timing.

*The timing routine was developed by Lt. E.A. Singer, a student at the Naval Postgraduate School.

VI. EFFICIENCY OF THE ALGORITHM

To eliminate the considerable time and effort required to input data by hand, a subroutine was designed which generates random problems of a large size. This routine uses a random number generator to produce the elements of the A, B, and C matrices. The following criteria were used in order to insure the existence of a bounded feasible solution:

maximize

$$z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j - x_{si} = b_i, \; i = 1,\ldots,m,$$

and

$$c_j \le 0, \; x_j \ge 0, \; x_{si} \ge 0, \; a_{ij} \ge 0, \; b_i \ge 0$$

$$\text{for } i = 1,\ldots,m, \; j = 1,\ldots,n,$$

where x_{si} is the slack variable for the i^{th} constraint.

For this investigation the subroutine generated problems having 70 variables (including 20 slack variables), 20 constraint equations, and 50 cost coefficients using the following uniform distributions:

a_{ij}, uniform (0,1);

b_i, uniform (0,5);

c_j, uniform (-1,0).

A total of 40 problems were solved using the random problem generator described above. The execution times for these problems are given in Table I at the end of this section.

The comparison of solution times shows that all 40 problems ran faster using the method described in this thesis, than with the standard primal-dual algorithm. The time differences range from 15.17 seconds to 48.18 seconds with an average time difference of 27.53 seconds.

TABLE I

Prob. No. i	Full Array Time (x_i) (sec.)	Full Array Iter.	Addition of Constraints Time (y_i) (sec.)	Addition of Constraints Iter	$x_i - y_i$
1	87.82	23	56.50	21	31.32
2	87.84	23	56.91	23	30.93
3	80.22	21	56.53	21	23.69
4	87.93	23	56.70	22	31.23
5	84.06	22	58.61	23	25.45
6	91.63	24	56.62	22	35.01
7	95.45	25	58.65	23	36.80
8	80.31	21	56.52	21	23.79
9	99.60	26	65.67	32	33.93
10	80.16	21	56.51	21	23.65
11	80.18	21	56.50	21	23.68
12	91.65	24	65.12	29	26.53
13	80.19	21	57.99	22	22.20
14	91.74	24	64.35	30	27.39
15	80.20	21	56.48	21	23.72
16	84.02	22	57.10	22	26.92
17	95.52	25	59.36	24	36.16
18	83.95	22	58.71	22	25.24
19	84.14	22	65.47	25	18.67
20	99.25	26	70.03	28	29.22
21	107.08	28	58.90	25	48.18
22	87.85	23	56.51	21	31.34
23	84.01	22	61.39	24	22.62
24	80.16	21	56.52	21	23.64
25	80.17	21	56.49	21	23.68
26	80.16	21	56.50	21	23.66
27	80.19	21	56.52	21	23.67
28	84.21	22	58.84	23	25.37
29	103.27	27	57.85	24	45.42
30	87.83	23	59.47	23	28.36
31	84.07	22	59.33	22	24.74
32	88.19	23	56.92	23	31.27
33	80.28	21	56.56	21	23.72
34	87.96	23	56.56	21	31.40
35	84.11	22	57.09	23	27.02
36	84.05	22	56.56	21	27.49
37	80.20	21	56.51	21	23.69
38	80.18	21	56.54	21	23.64
39	84.03	22	68.86	26	15.17
40	80.29	21	56.54	21	23.75
Total	3454.01		2352.77		1101.24
Average	86.35		58.82		27.53

VII. SUMMARY AND CONCLUSIONS

A modification of the primal-dual algorithm has been presented. This modification differs from the standard primal-dual algorithm in that the constraint equations are introduced one at a time, and each subproblem is solved before the next constraint is added.

This algorithm was programmed in FORTRAN IV for the IBM 360/67. The program was designed so that any given linear program is solved first by the standard primal-dual algorithm, and then is resolved using the modified primal-dual procedure. Further, the same subroutines are used for both methods in order to eliminate timing bias due to coding differences. In fact the modified procedure contains steps which are not included in the timing of the standard routine.

In all cases the modified procedure was faster than the standard procedure. Of 40 test problems the standard method averaged 86.35 seconds per problem, whereas the modified method averaged 58.82 seconds per problem. One should not judge the actual running time of the test problems since no attempt was made to improve the efficiency of the computer program on an absolute basis; only the relative speeds of the two methods is of importance.

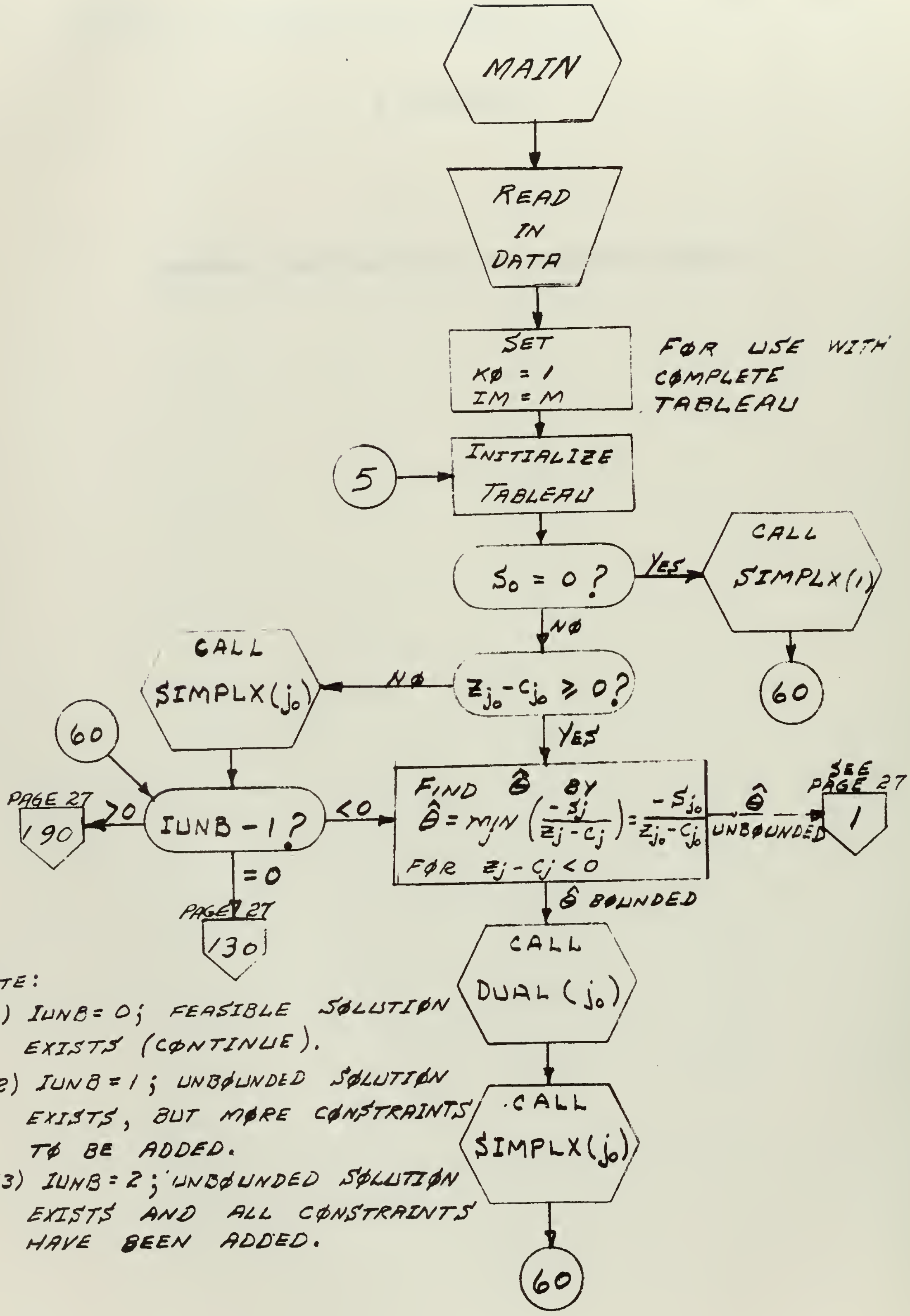
BIBLIOGRAPHY

1. Hadley, G., Linear Programming. Reading, Mass: Addison-Wesley Publishing Co., Inc., 1962.

APPENDIX A

FLOW DIAGRAMS OF THE COMPUTER PROGRAM

1. MAIN PROGRAM

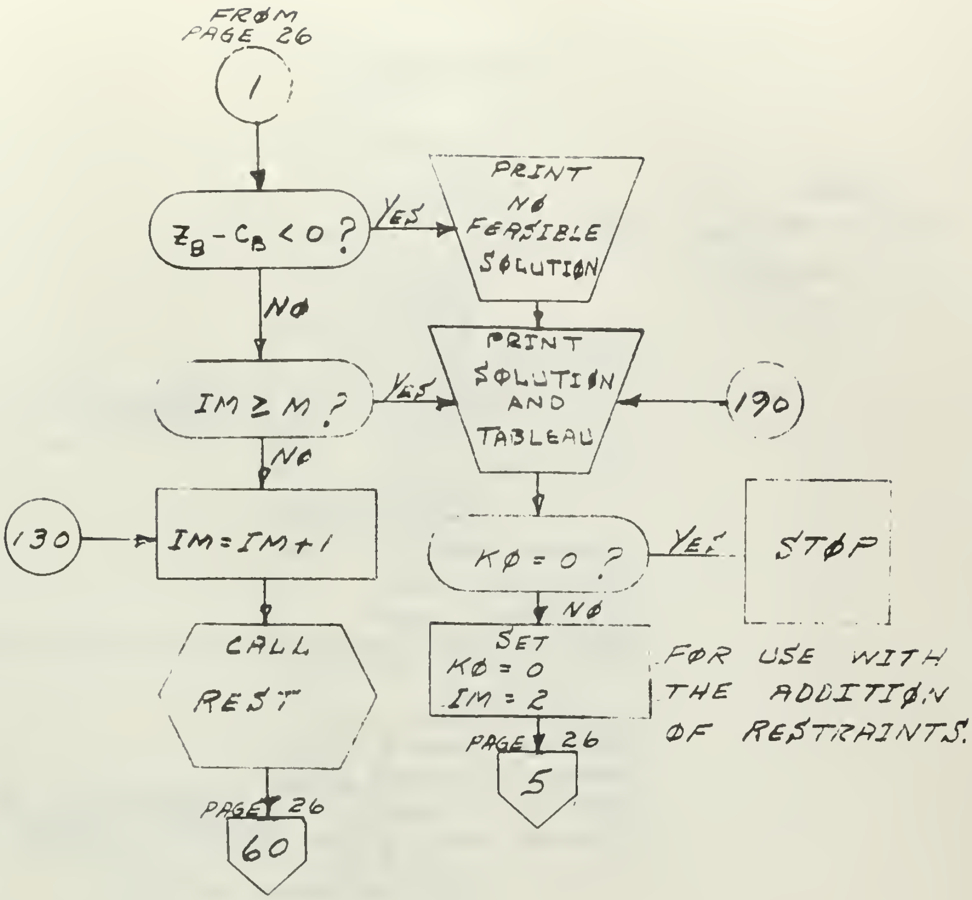


NOTE:

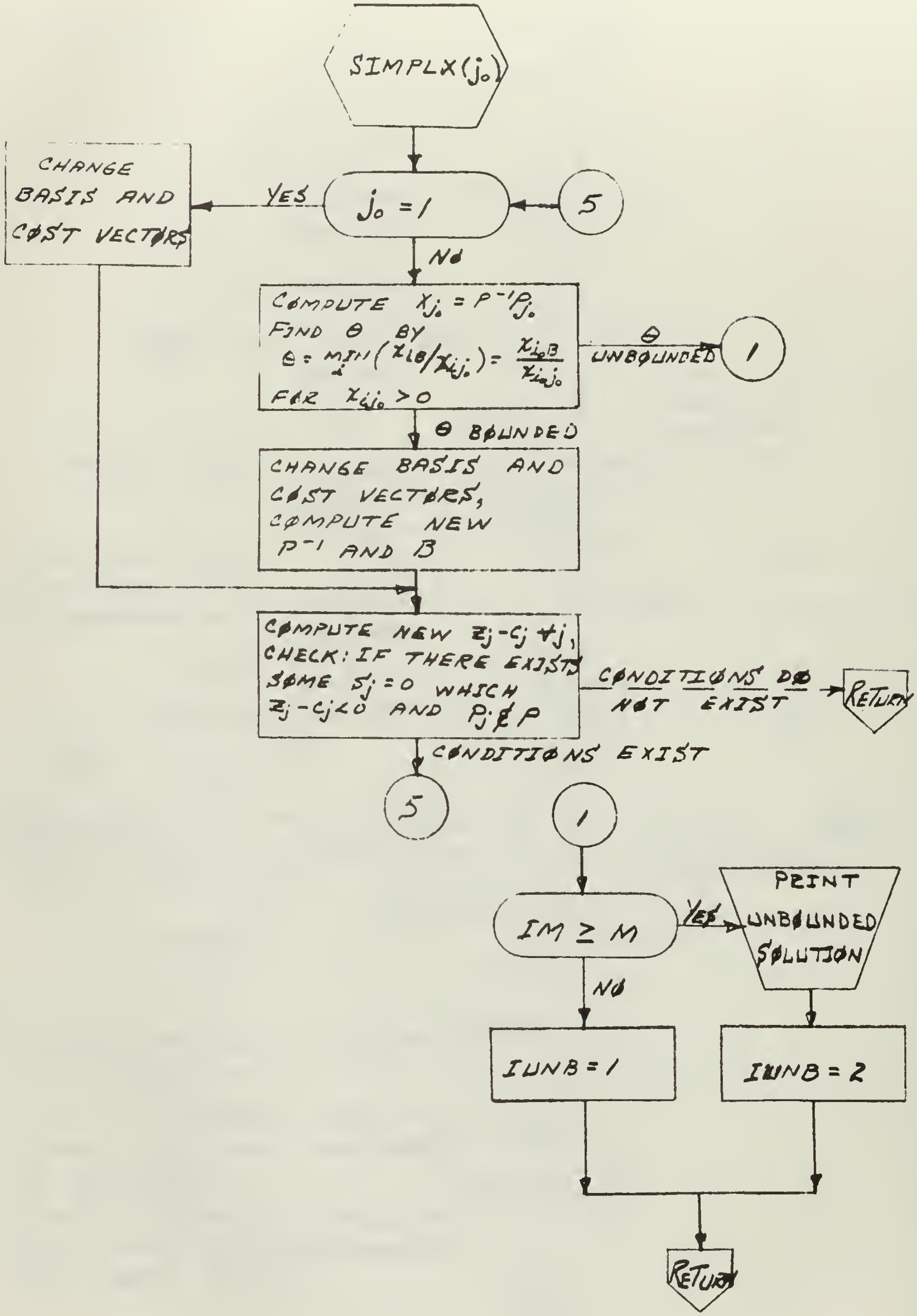
(1) IUNB = 0; FEASIBLE SØLUTIØN EXISTS (CØNTINUE).

(2) IUNB = 1; UNBØUNDED SØLUTIØN EXISTS, BUT MØRE CØNSTRAINTS TØ BE ADDED.

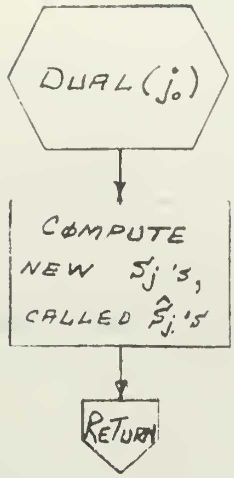
(3) IUNB = 2; UNBØUNDED SØLUTIØN EXISTS AND ALL CØNSTRAINTS HAVE BEEN ADDED.


FROM PAGE 26
1
$Z_B - C_B < 0$?
YES
PRINT NO FEASIBLE SOLUTION
NO
IM ≥ M ?
YES
PRINT SOLUTION AND TABLEAU
190
NO
130
IM=IM+1
KO = 0 ?
YES
STOP
NO
CALL REST
SET KO = 0 IM = 2
FOR USE WITH THE ADDITION OF RESTRAINTS.
PAGE 26
5
PAGE 26
60

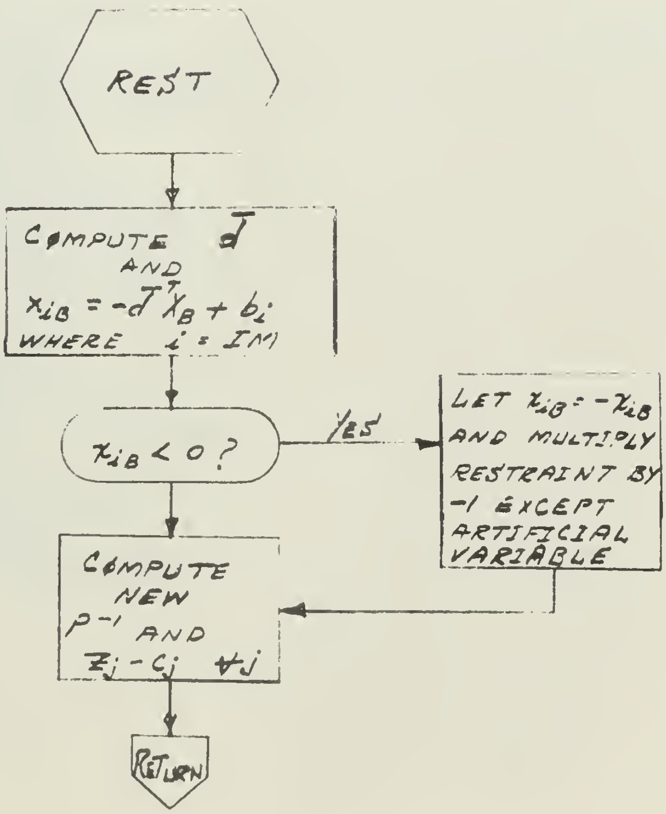
2. SIMPLEX ITERATION SUBROUTINE



3. DUAL ITERATION SUBROUTINE



4. ADDITION OF NEW RESTRAINT SUBROUTINE



APPENDIX B

FORTRAN LISTING OF THE COMPUTER PROGRAM

```
     0DIMENSION P(53,151),B(52),C(151),BT(53,2),IBAS(53),ONE(53),TEM(2),
     1PINV(53,53),ZCB(2),ZC(151),WSB(2),WS(151),T(2),X(53),DBAR(53)
      COMMON/INT/IUNB,IM,M,N,IBAS,ITER
      COMMON/FLOAT/P,B,C,BT,ONE,TEM,PINV,ZCB,ZC,WSB,WS,T,X,DBAR,EP
C
C NOTE - WS(J) IN THIS PROGRAM AND S(J) IN THESIS ARE EQUIVALENT
C
      EP = 10.0E-6
      XE = URN(0)
      IPROB = 0
    6 IF(IPROB.GE.50) GO TO 500
      IPROB = IPROB +1
      CALL PSEUDO
      M = M + 1
      N = N + 1
C
C KO=1 IS FOR USE OF COMPLETE TABLEAU
C
      KO = 1
      IM = M
C
C INITIALIZE TABLEAU
C
    5 DO 1 I=2,M
      BT(I,1) = 0.0
      BT(I,2) = B(I-1)
    1 P(I,1) = 0.0
      ITER = 1
      DO 2 J=1,N
    2 P(1,J) = 1.0
      BT(1,1) = 1.0
      BT(1,2) = 0.0
      DO 3 I=1,M
      IBAS(I) = N+I
    3 ONE(I) = -1.0
      DO 10 I=1,M
      DO 10 J=1,M
      IF(I.EQ.J) GO TO 9
      PINV(I,J) = 0.0
      GO TO 10
    9 PINV(I,J) = 1.0
   10 CONTINUE
      CALL TIMEIT(0,CL)
      ZCB(1) = -1.0
      ZCB(2) = 0.0
      DO 11 I=2,IM
   11 ZCB(2) = ZCB(2) -BT(I,2)
      ZC(1) = -1.0
      DO 20 J =2,N
      ZC(J) = -1.0
      DO 20 I = 2,IM
   20 ZC(J) = ZC(J) -P(I,J)
      IND = 0
      WS(1) = 0.0
      DO 30 J=2,N
      IF(C(J).LE.WS(1))GO TO 30
      WS(1) = C(J)
      IND = J
   30 CONTINUE
      WSB(1) = WS(1)
      WSB(2) = 0.0
      DO 40 J=2,N
   40 WS(J) = WS(1) -C(J)
C
C DETERMINE VECTOR TO INTRODUCE
C
      IF(IND.EQ.0.0)GO TO 50
      IF(ZC(IND).GE.0.0)GO TO 70
      CALL SIMPLX(IND)
      GO TO 60
   50 CALL SIMPLX(1)
   60 IF(IUNB-1)70,130,190
   70 TEMP = 999999.
```

```
C
C  FIND THETA FOR DUAL = MIN(-S(J)/ZC(J))OVER J FOR ZC(J)<0
C
      IND = 0
      DO 80 J=1,N
      IF(ZC(J).GE.0.0)GO TO 80
      DO 75 I=1,IM
      IF(IBAS(I).EQ.J) GO TO 80
   75 CONTINUE
      XE= -WS(J)/ZC(J)
      IF(TEMP.LE.XE)GO TO 80
      TEMP = XE
      IND = J
   80 CONTINUE
      IF(IND.EQ.0) GO TO 100
      CALL DUAL(IND)
      CALL SIMPLX(IND)
      GO TO 60
C
C  CHECK FOR INFEASIABILITY
C
  100 IF(ZCB(1))200,110,120
  110 IF(ZCB(2).LT.0.0) GO TO 200
  120 IF(IM.GE.M) GO TO 190
C
C  ADD NEXT RESTRAINT
C
  130 IM = IM + 1
      CALL REST
      IUNB = 0
      GO TO 60
  190 CALL TIMEIT(-1,CL)
      WRITE(6,5000)IPROB,CL,ITER
 5000 FORMAT(10X,'PROBLEM',I4,'  TIME IS ',-6PF15.6,'  SECONDS WITH',
     1I6,'  ITERATIONS',//)
      IF(KO.EQ.0)GO TO 6
C
C  KO=0 IS FOR USE OF ADDITION OF RESTRAINTS
C
      IM=2
      KO=0
      GO TO 5
  500 STOP
  200 WRITE(6,4010)
      GO TO 190
 4010 FORMAT(10X,'SOLUTION INFEASIBLE'//)
      END
```

```
          C
          C
          C
0001            SUBROUTINE SIMPLX(J)                                              0001090
0002           0DIMENSION P(53,151),B(52),C(151),BT(53,2),IBAS(53),ONE(53),TEM(2),  0001100
               1PINV(53,53),ZCB(2),ZC(151),WSB(2),WS(151),T(2),X(53),DBAR(53)     0001110
0003            COMMON/INT/IUNB,IM,M,N,IBAS,ITER                                  0001120
0004            COMMON/FLOAT/P,B,C,BT,ONE,TEM,PINV,ZCB,ZC,WSB,WS,T,X,DBAR,EP      0001125
0005            IUNB = 0                                                          0001130
          C
          C FOR J=1 THE ONLY CHANGE IS THE BASIS AND ZC(J)'S
          C
0006          5 IF(J.EQ.1) GO TO 130                                              0001140
          C
          C COMPUTE X(J) AND FIND THETA FOR PRIMAL = MIN(X(I,B)/X(I,J)) OVER J
          C    FOR X(I,J)>0
          C
0007            T(1) = 999999.                                                    0001150
0008            T(2) = T(1)                                                       0001160
0009            DO 20 I=1,IM                                                      0001170
0010            L = IM-I+1                                                        0001180
0011            X(L) = 0.0                                                        0001190
0012            DO 10 K=1,IM                                                      0001200
0013            Y = PINV(L,K)*P(K,J)                                              0001205
0014            X(L) = X(L) + Y                                                   0001210
0015         10 CALL ROUND(X(L),Y,EP)                                             0001211
0016            IF(X(L).LE.0.)GO TO 20                                            0001215
0017            TEM(1) = BT(L,1)/X(L)                                             0001220
0018            TEM(2) = BT(L,2)/X(L)                                             0001230
0019            IF(T(1)-TEM(1))20,11,12                                           0001240
0020         11 IF(T(2).LE.TEM(2))GO TO 20                                        0001250
0021         12 T(1) = TEM(1)                                                     0001260
0022            T(2) = TEM(2)                                                     0001270
0023            ID = L                                                            0001280
0024         20 CONTINUE                                                          0001290
          C
          C COMPUTE NEW TABLEAU
          C
0025            IF(T(1).EQ.999999.)GO TO 50                                       0001300
0026            IBAS(ID) = J                                                      0001310
0027            ONE(ID) = 0.0                                                     0001320
0028            DO 30 I=1,IM                                                      0001330
0029            IF(I.EQ.ID) GO TO 30                                              0001333
0030            Y = T(1)*X(I)                                                     0001333
0031            BT(I,1) = BT(I,1) - Y                                             0001334
0032            CALL ROUND(BT(I,1),Y,EP)                                          0001335
0033            Y = T(2)*X(I)                                                     0001336
0034            BT(I,2) = BT(I,2) - Y                                             0001336
0035            CALL ROUND(BT(I,2),Y,EP)                                          0001337
0036            XE = X(I)/X(ID)                                                   0001360
0037            DO 30 K=1,IM                                                      0001370
0038            Y = PINV(ID,K)*XE                                                 0001371
0039            PINV(I,K) = PINV(I,K) - Y                                         0001372
0040            CALL ROUND(PINV(I,K),Y,EP)                                        0001375
0041         30 CONTINUE                                                          0001395
0042            BT(ID,1) = BT(ID,1)/X(ID)                                         0001387
0043            BT(ID,2) = BT(ID,2)/X(ID)                                         0001388
0044            DO 31 I=1,IM                                                      0001386
0045         31 PINV(ID,I) = PINV(ID,I)/X(ID)                                     0001389
0046         70 ZCB(1) = 0.0                                                      0001390
0047            ZCB(2) = 0.0                                                      0001410
0048            DO 80 I=1,IM                                                      0001420
0049            ZCB(1) =ZCB(1) +ONE(I)*BT(I,1)                                    0001430
0050         80 ZCB(2) =ZCB(2) +ONE(I)*BT(I,2)                                    0001440
0051            ZC(1) = ONE(1)                                                    0001450
0052            DO 90 L=2,N                                                       0001460
0053            ZC(L) = 0.                                                        0001470
0054            DO 90 I=1,IM                                                      0001490
0055            DO 90 K=1,IM                                                      0001500
0056            Y = ONE(I)*PINV(I,K)*P(K,L)                                       0001505
0057            ZC(L) = ZC(L) + Y                                                 0001510
0058         90 CALL ROUND(ZC(L),Y,EP)                                            0001515
          C
```

```
C CHECK FOR AN ADDITION ITERATION WITH PRIMAL
C
      TEMP = 0.0
      DO 120 L=1,N
      IF(WS(L).NE.0.0) GO TO 120
      DO 110 I=1,IM
      IF(IBAS(I).EQ.L)GO TO 120
  110 CONTINUE
      IF(ZC(L).GE.TEMP) GO TO 120
      J=L
      TEMP = ZC(L)
  120 CONTINUE
      ITER = ITER + 1
      IF(TEMP.NE.0.0) GO TO 5
  200 RETURN
  130 IBAS(1) = 1
      ONE(1) = 0.0
      ID = 1
      GO TO 70
C
C SOLUTION UNBOUNDED
C
   50 IF(IM.GE.M) GO TO 60
      IUNB = 1
      GO TO 200
   60 WRITE(6,4020)
      IUNB = 2
 4020 FORMAT(10X,'SOLUTION UNBOUNDED'//)
      GO TO 200
      END
```

```
              C
              C
              C
0001                SUBROUTINE DUAL(J)                                                   0001750
0002               0DIMENSION P(53,151),B(52),C(151),BT(53,2),IBAS(53),ONE(53),TEM(2),   0001760
                   1PINV(53,53),ZCB(2),ZC(151),WSB(2),WS(151),T(2),X(53),DBAR(53)        0001770
0003                COMMON/INT/IUNB,IM,M,N,IBAS,ITER                                     0001780
0004                COMMON/FLOAT/P,B,C,BT,ONE,TEM,PINV,ZCB,ZC,WSB,WS,T,X,DBAR,EP         0001785
              C
              C COMPUTE NEW WS(J) WHICH IS EQUIVALENT TO S(J)
              C
0005                XE = WS(J)/ZC(J)                                                     0001790
0006                DO 10 K=1,N                                                          0001800
0007                Y = ZC(K)*XE                                                         0001801
0008                WS(K) = WS(K) - Y                                                    0001802
0009             10 CALL ROUND(WS(K),Y,EP)                                               0001803
0010                Y = ZCB(1)*XE                                                        0001804
0011                WSB(1) = WSB(1) - Y                                                  0001805
0012                CALL ROUND(WSB(1),Y,EP)                                              0001806
0013                Y = ZCB(2)*XE                                                        0001807
0014                WSB(2) = WSB(2) - Y                                                  0001808
0015                CALL ROUND(WSB(2),Y,EP)                                              0001810
0016                RETURN                                                               0001840
0017                END                                                                  0001850
```

```
      C
      C
      C
            SUBROUTINE REST                                                    0001860
           0DIMENSION P(53,151),B(52),C(151),BT(53,2),IBAS(53),ONE(53),TEM(2), 0001870
           1PINV(53,53),ZCB(2),ZC(151),WSB(2),WS(151),T(2),X(53),DBAR(53)     0001880
            COMMON/INT/IUNB,IM,M,N,IBAS,ITER                                   0001890
            COMMON/FLOAT/P,B,C,BT,ONE,TEM,PINV,ZCB,ZC,WSB,WS,T,X,DBAR,EP       0001895
      C
      C COMPUTE D-BAR AND B VECTOR
      C
            K = IM-1                                                           0001900
            BT(IM,1) = 0.0                                                     0001910
            BT(IM,2) = B(K)                                                    0001920
            DO 10 I=1,K                                                        0001930
            IF(IBAS(I).GT.N) GO TO 5                                           0001935
            L = IBAS(I)                                                        0001940
            DBAR(I) = -P(IM,L)                                                 0001950
            BT(IM,1) = BT(IM,1) + DBAR(I)*BT(I,1)                              0001960
            BT(IM,2) = BT(IM,2) + DBAR(I)*BT(I,2)                              0001970
            GO TO 10                                                           0001971
          5 DBAR(I) = 0.0                                                      0001972
         10 CONTINUE                                                           0001973
            IF(BT(IM,1))50,20,30                                               0001980
         20 IF(BT(IM,2).LT.0.0) GO TO 50                                       0001990
      C
      C COMPUTE NEW INVERSE AND COST VECTOR
      C
         30 DO 40 J=1,K                                                        0002000
            DO 40 I=1,K                                                        0002010
         40 PINV(IM,J) = PINV(IM,J) + DBAR(I)*PINV(I,J)                        0002020
            ZC(1) = ONE(1)                                                     0002021
            DO 45 J=2,N                                                        0002022
            ZC(J) =0.0                                                         0002023
            DO 45 I=1,IM                                                       0002024
            DO 45 IK=1,IM                                                      0002025
            Y = ONE(I)*PINV(I,IK)*P(IK,J)                                      0002026
            ZC(J) = ZC(J) + Y                                                  0002027
         45 CALL ROUND(ZC(J),Y,EP)                                             0002028
            ZCB(1) = ZCB(1) - BT(IM,1)                                         0002029
            ZCB(2) = ZCB(2) - BT(IM,2)                                         0002030
        100 RETURN                                                             0002031
      C
      C USED TO INSURE FEASIABILITY BY MAKING THE B COMPONENT NON-NEGATIVE
      C
         50 BT(IM,1) = -BT(IM,1)                                               0002040
            BT(IM,2) = -BT(IM,2)                                               0002050
            DO 60 I=1,K                                                        0002060
         60 DBAR(I) = -DBAR(I)                                                 0002070
            PINV(IM,IM) = -1.0                                                 0002130
            GO TO 30                                                           0002140
            END                                                                0002150
```

```
      C
      C
      C
            SUBROUTINE ROUND(A,B,EP)
      C  A=STARTING VALUE, B=ADDED VALUE, EP= LOWEST ROUND-OFF DESIRED
            IF(A.EQ.0.0.OR.B.EQ.0.0) GO TO 200
            IF(ABS(A/B).GT.EP) GO TO 200
            A = 0.0
        200 RETURN
            END
```

```
              C
              C
              C
 0001               SUBROUTINE TIMEIT(N,TIME)                                          0002230
              C
              C  N=0 STARTS CLOCK, N=-1 STOPS CLOCK
              C
 0002               IT=N+2                                                             0002270
 0003               GO TO (20,10),IT                                                   0002280
 0004            10 CALL TIMON(M)                                                      0002290
 0005               TIMEM=M                                                            0002300
 0006               RETURN                                                             0002310
 0007            20 CALL TIMOFF(M)                                                     0002320
 0008               TIME=M                                                             0002330
 0009               TIME=(TIMEM-TIME)*26.0                                             0002340
 0010               RETURN                                                             0002350
 0011               END                                                                0002360
```

```
      C
      C
      C
            SUBROUTINE PSEUDO
           0DIMENSION P(53,151),B(52),C(151),BT(53,2),IBAS(53),ONE(53),TEM(2),
           1PINV(53,53),ZCB(2),ZC(151),WSB(2),WS(151),T(2),X(53),DBAR(53)
            COMMON/INT/IUNB,IM,M,N,IBAS,ITER
            COMMON/FLOAT/P,B,C,BT,ONE,TEM,PINV,ZCB,ZC,WSB,WS,T,X,DBAR,EP
      C
      C M = NUMBER OF CONSTRAINTS, N = NUMBER OF VARIABLES INCLUDING M SLACK
      C    VARIABLES
      C MAXIMUM M = 50, MAXIMUM N = 150
      C
      C CHANGE STATEMENTS 1 AND 2 TO DESIRED PROBLEM SIZE
      C
          1 M = 20
          2 N = 70
            IM = M + 1
            IN = N + 1
            K1 = N - M + 1
            K2 = K1 + 1
            C(J) = -URN(1)
            DO 20 J=2,K
            IF(C(J).LE.-.001) GO TO 10
            C(J) = 0.0
         10 DO 20 I=2,IM
            P(I,J) = URN(1)
            IF(P(I,J).GT.0.001) GO TO 20
            P(I,J) = 0.0
         20 CONTINUE
            DO 30 I=1,M
            B(I) = URN(1)*5.0
            IF(B(I).GT.0.01) GO TO 30
            B(I) = 0.0
         30 CONTINUE
            DO 40 J=K2,IN
            C(J) =0.0
            DO 40 I = 2,IM
         40 P(I,J) = 0.0
            DO 50 I=2,IM
            K = N- M + I
         50 P(I,K) = -1.0
            RETURN
            END
```

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13. ABSTRACT

Linear programming techniques are becoming of greater importance because the use of computerization has increased the fields for applications for linear programs. The primal-dual algorithm, in which the constraints are added one at a time, is investigated as a possible faster solution method. A computer program was developed to compare this method with the standard primal-dual algorithm using the full set of constraints at one time. Several random problems were solved using these two methods, and the results indicated a significant improvement in the solution time by the use of adding the constraints one at a time.

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PRIMAL-DUAL ALGORITHM						

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